

Two-colorable graph states with maximal Schmidt measure

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The Schmidt measure was introduced by Eisert and Briegel for quantifying the degree of entanglement of multipartite quantum systems [*Phys. Rev. A* **64**, 022306 (2001)]. Although generally intractable, it turns out that there is a bound on the Schmidt measure for two-colorable graph states [*Phys. Rev. A* **69**, 062311 (2004)]. For these states, the Schmidt measure is in fact directly related to the number of nonzero eigenvalues of the adjacency matrix of the associated graph. We remark that almost all two-colorable graph states have maximal Schmidt measure and we construct specific examples. These involve perfect trees, line graphs of trees, cographs, graphs from anti-Hadamard matrices, and unicyclic graphs. We consider some graph transformations, with the idea of transforming a two-colorable graph state with maximal Schmidt measure into another one with the same property. In particular, we consider a transformation introduced by François Jaeger, line graphs, and switching. By making appeal to a result of Ehrenfeucht *et al.* [*Discrete Math.* **278** (2004)], we point out that local complementation and switching form a transitive group acting on the set of all graph states of a given dimension.

Keywords: graph states, Schmidt measure, local complementation, switching, nonsingular graphs, trees, line graphs, cographs, unicyclic graphs.

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I. INTRODUCTION

Graph states are certain pure multi-party quantum states associated to graphs (see, *e.g.*, [32, 35, 36]). Graph states have important applications in quantum error correction and in the one-way quantum computer (see [24] and [4], respectively). The graph states associated to bipartite graphs, also called *two-colorable graph states* [2], have useful properties. For example, Greenberger-Horne-Zeilinger states and cluster states are two-colorable graph states [27]. In addition, two-colorable graph states are equivalent (up to local unitary transformations) to Calderbank-Shor-Steane states [16].

The Schmidt measure, based on a generalization of the Schmidt rank of pure states, was introduced by Eisert and Briegel [14] for quantifying the degree of entanglement of multipartite quantum systems. Although generally intractable, it turns out that it is easy to bound the Schmidt measure for two-colorable graph states. The Schmidt measure is in fact directly related to the rank of the graph associated to the state. (It has to be remarked that an entanglement measure for graph states, which can be computed efficiently from a set of generators of the stabilizer group, was introduced in [18].) The *rank of a graph* is the number of nonzero eigenvalues of the adjacency matrix; its study has a number of applications [10]. For example, in chemistry, the graph representing the carbon-atom skeleton of a molecule has full rank (that is, equal to the number of vertices) if the so-called conjugated molecule is chemically stable.

This paper presents a portfolio of two-colorable graph states with maximal Schmidt measure. Practically, this

paper mainly surveys results on $(0,1)$ invertible matrices and merely translates these into the context of graph states. The paper is organized as follows. In Section 2, we recall the notion of graph state and Schmidt measure. In Section 3, we describe some classes of graphs with full rank. These concern perfect trees, certain bipartite graphs obtained from a class of cographs (called here Bıyıkoglu cographs), certain graphs obtained from anti-Hadamard matrices, and unicyclic graphs. In Section 4, we consider some graph transformations, with the idea of transforming a two-colorable graph state with maximal Schmidt measure into another one with the same property. In particular, we consider an operation introduced by François Jaeger, line graphs, and switching. Even if it does not seem easy to establish some general relation between switching and the rank of the adjacency matrix, we claim that switching is worth further study. In fact, together with local complementation (a graph operation which is central in the context of graph states), it generates a transitive group acting on the set of graph states of a given dimension.

II. GRAPH STATES

Firstly, let us recall the definition of a graph state [36]. A *graph* $G = (V(G), E(G))$ is a pair whose elements are two set, $V(G) = \{1, 2, \dots, n\}$ and $E(G) \subset V(G) \times V(G)$. The elements of $V(G)$ and $E(G)$ are called *vertices* and *edges*, respectively. We assume that $\{i, i\} \notin E(G)$ for all $i \in V(G)$. The *adjacency matrix* of G is the matrix $A(G)$ such that $A(G)_{i,j} = 1$ if $\{i, j\} \in E(G)$ and $A(G)_{i,j} = 0$ if $\{i, j\} \notin E(G)$. The matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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are called *Pauli matrices*. Let I be the identity matrix. Given a graph G , we define a block-matrix $S(G)$ with ij -th block defined as follows:

$$S(G)_{i,j} = \begin{cases} \sigma_x, & \text{if } i = j; \\ \sigma_z, & \text{if } \{i, j\} \in E(G); \\ I, & \text{if } \{i, j\} \notin E(G). \end{cases}$$

From the block-rows of $S(G)$ we construct the following matrices:

$$\begin{aligned} S(G)_1 &= S(G)_{1,1} \otimes S(G)_{1,2} \otimes \cdots \otimes S_{1,n}, \\ S(G)_2 &= S(G)_{2,1} \otimes S(G)_{2,2} \otimes \cdots \otimes S_{2,n}, \\ &\vdots \\ S(G)_n &= S(G)_{n,1} \otimes S(G)_{n,2} \otimes \cdots \otimes S_{n,n}. \end{aligned}$$

It can be shown that these matrices all commute. The *graph state* associated to the graph G is defined to be the common eigenvector of the matrices $S(G)_1, S(G)_2, \dots, S(G)_n$ with eigenvalue 1. We denote by $|G\rangle$ the graph state corresponding to the graph G .

Let $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ be an Hilbert space assigned to a quantum system with n subsystems. The pure state $|\psi\rangle$ of the system can be written as

$$|\psi\rangle = \sum_{i=1}^R \alpha_i |\psi_i\rangle_1 \otimes |\psi_i\rangle_2 \otimes \cdots \otimes |\psi_i\rangle_n, \quad (1)$$

where $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \dots, R$, and $|\psi_i\rangle_j \in \mathcal{H}_j$ for $j = 1, 2, \dots, n$. The *Schmidt measure* of $|\psi\rangle$ is defined by $E_S(|\psi\rangle) = \log_2(r)$, where r is the minimal number R of terms in the summation of Eq. (1) over all linear decompositions into product states.

Two vertices i, j of a graph are said to be *adjacent* if $\{i, j\}$ is an edge (and the edge is then *incident* with the vertices). A graph is *bipartite* if it has a bipartition of the set of vertices into two disjoint sets where vertices in one set are adjacent only to vertices in the other set. The *spectrum* of a graph G is the collection of eigenvalues of $M(G)$, or equivalently, the collection of zeros of the characteristic polynomial of $M(G)$ (see, e.g., [10]). The *rank* of G , denoted by $r(G)$, is the number of non-zero eigenvalues in the spectrum of $M(G)$. The *nullity* of G is the number of eigenvalues of $M(G)$ which are equal to zero. A number of important parameters of a graph G is bounded by a function of $r(G)$, for example, clique number, chromatic number, etc. The rank of G is bounded above by the number of distinct nonzero rows of $M(G)$. A graph is *nonsingular* if $r(G) = |V(G)|$. The following proposition gives bounds on the Schmidt measure of graph states associated to bipartite graphs:

Proposition 1 ([14]) *Let G be a bipartite graph. Then*

$$\frac{1}{2}r(M(G)) \leq E_S(|G\rangle) \leq \left\lfloor \frac{|V|}{2} \right\rfloor.$$

Moreover, if $r(G) = n$ then $E_S(|G\rangle) = \left\lfloor \frac{|V|}{2} \right\rfloor$.

Before to move on to the next sections, it is useful to describe a way to construct a bipartite graph on $2n$ vertices from an $n \times n$ $(0, 1)$ -matrix (recall that a $(0, 1)$ -matrix is a matrix whose entries are in the set $\{0, 1\}$). Let G be a graph with adjacency matrix

$$M(G) = P \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix} P^T,$$

where P is some permutation matrix and M is a $(0, 1)$ -matrix. It is clear that G is bipartite and that the rank of G is twice the rank of M . We say that G is the *bipartite double* of the (possibly directed) graph with adjacency matrix M .

III. BIPARTITE GRAPHS WITH MAXIMAL RANK

By Proposition 1, a graph state $|G\rangle$ has maximal Schmidt measure if the bipartite graph G is nonsingular. In the next subsections we present (in this order) the following bipartite nonsingular graphs:

- perfect trees;
- Büyükoğlu cographs;
- graphs from anti-Hadamard matrices;
- certain unicyclic graphs.

A. General remarks

Minimum number of edges. A *path* in a graph is a finite sequence of alternating vertices and edges, starting and ending with a vertex, $v_1 e_1 v_2 e_2 v_3 \dots e_{n-1} v_n$, such that every consecutive pair of vertices v_x and v_{x+1} are adjacent and e_x is incident with v_x and with v_{x+1} . The *length* of a path is the number of its vertices. A path of length n is denoted by P_n . A *cycle* of length $n - 1$, that is a path in which $v_1 = v_n$, is denoted by C_{n-1} . A *connected graph* is a graph such that there is a path between all pairs of vertices. A *connected component* is a maximal subset of vertices and edges between them that forms a connected graph. On the base of [1] (Proposition 18), we can write:

Proposition 2 *The smallest number of edges of a graph on n vertices associated to a two-colorable graph state with maximal Schmidt measure is $n/2$ if n is even and $(n + 3)/2$ if n is odd; if the graph is connected then the numbers are $n - 1$ if n is even and n if n is odd. (The graphs in question are obvious.)*

Random states. Numerical evidence gives strong support to the conjecture that the probability that an $n \times n$ $(0, 1)$ -matrix is singular is exactly $n^2 2^{-n}$ (see, e.g., [29, 41]). Clearly, $\lim_{n \rightarrow \infty} n^2 2^{-n} = 0$. Based on this conjecture, it is relatively safe to believe that *almost all*

two-colorable graph states have maximal Schmidt measure.

Nullity preserving operations. When a vertex and its incident edges are deleted from a graph G , the rank of the resultant graph cannot exceed $r(G)$ and can decrease by at most 2. When edges are added to a graph, the rank of the resultant graph cannot decrease and can increase by at most 2 [3]. The following two operations preserve the nullity of a graph [10]: (i) A path of length 6 is replaced by an edge. (ii) A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and every arc in $E(H)$ has both its end-vertices in $V(G)$. A subgraph H of G is an *induced subgraph* if every edge in $E(G)$, having both vertices in $V(H)$, is also in $E(H)$. For a graph G having vertex incident with one edge only, the induced subgraph H , obtained by deleting this vertex together with the vertex adjacent to it, has the same nullity of G .

B. Perfect trees

A *tree* is a graph with no path that starts and ends at the same vertex. A *matching* in a graph G is a set $S \subseteq E(G)$ no two edges are incident with a common vertex. A matching S is *perfect* if $|V| = 2|S|$. The *matching number*, denoted by $\beta(G)$, is the largest cardinality of a matching in G . For some classes of graphs (for example, trees) $r(G)$ and $\beta(G)$ are related. A *complete graph* on n vertices, denoted by K_n , is a graph such that $E(K_n) = V(K_n) \times V(K_n)$. A *perfect tree* is defined as follows: the tree K_2 is perfect; if T_1 and T_2 are perfect trees then the tree obtained by adding an edge between any vertex of T_1 and any vertex of T_2 is also perfect.

The authors of [27] observed that the Schmidt measure of a tree can be obtained from the size of its smallest vertex cover (that is, the minimum number of vertices required to cover all edges). Let T be a tree on n vertices. It is well-known that (see, e.g., [4] or [5], Theorem 8.1) $r(T) = 2 \cdot \beta(T)$. Then $r(T) = n$ if and only if T has a perfect matching. One can show that a tree has a perfect matching if and only if it is *perfect* (a proof of this result is Lemma 3.2 in [20]). Given that a graph is bipartite if and only if it has no cycles of odd length, a tree is bipartite since it is no cycles by definition. The following is a direct consequence of this reasoning:

Proposition 3 *Let T be a perfect tree on n vertices. Then $E_S(|T\rangle)$ is maximal.*

C. Cographs

A graph G is said to be *H-free* if H is not an induced subgraph of G . A *cograph* is a P_4 -free graph. An important subclass of cographs consists of threshold graphs. These are applied to integer programming, synchronization of parallel processes, etc. (see, e.g., [21]). A connected cograph is bipartite if and only if it is a complete

bipartite graph, and cographs on n vertices and m edges are recognized in time $O(n + m)$. A simple construction of cographs is given by Lovasz [30]. For a cograph G on n vertices, Royle [34] and Bıykoğlu [6] settled in the affirmative a conjecture of Sillke [38], by proving that if the rows of $M(G)$ are distinct and nonzero then $r(G) = n$. The cographs with invertible adjacency matrix have been characterized by Bıykoğlu (see [6], Lemma 4). For this reason, we refer to these graphs as *Bıykoğlu cographs*. Defining Bıykoğlu cographs is lengthy and it requires a number of notions. The interested reader is addressed to [6].

Proposition 4 *Let G be the bipartite double of a Bıykoğlu cograph. Then $E_S(|G\rangle)$ is maximal.*

D. Graphs from anti-Hadamard matrices

An *anti-Hadamard matrix* M is an $n \times n$ $(0, 1)$ -matrix for which $\mu(M) = \mu(n)$, where $\mu(M) = \sum_{i,j=1}^n (M_{i,j}^{-1})^2$ (the square of the Euclidean norm of M^{-1}) and $\mu(n) = \max_M \mu(M)$, with the maximum taken over all invertible $(0, 1)$ -matrices. Anti-Hadamard matrices were introduced by Graham and Sloane as $(0, 1)$ -matrices which are “only just nonsingular” [22]. Next is an anti-Hadamard matrix:

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Proposition 5 *Let G be the bipartite double of a digraph whose adjacency matrix is an anti-Hadamard matrix. Then $E_S(|G\rangle)$ is maximal.*

E. Unicyclic graphs

A graph is *unicyclic* if it has the same number of vertices and edges. A unicyclic graph G satisfying one of the two following two properties is said to be *elementary*: (i) The graph G is the cycle C_n where $n \not\equiv 0 \pmod{4}$ (ii) The graph G is constructed as follows: select t vertices from C_l such that between two selected vertices there is an even (possibly 0) number of vertices, and t is an integer such that $0 < t \leq l$ with $l \equiv t \pmod{2}$; each one of these selected t vertices is then joined to one of t extra vertices. By a result of [42], we have the next fact:

Proposition 6 *If G is an elementary unicyclic graph, or a graph obtained by joining a vertex of a perfect tree with an arbitrary vertex of an elementary unicyclic graph, then $E_S(|G\rangle)$ is maximal.*

IV. GRAPH TRANSFORMATIONS

A. Inverse of the adjacency matrix

Local complementation is a graph transformation whose study was principally carried on by Bouchet [4] and Fon-der-Flaas [19]. Local complementation is important in the context of graph states. Given a graph G , the *neighborhood* of $i \in V(G)$ is $N(i) = \{j : \{i, j\} \in E(G)\}$. The graph G_i^c is the *local complement* of G at i if $V(G_i^c) = V(G)$ and $\{k, l\} \in E(G_i^c)$ if and only if one of the following two conditions is satisfied: (i) $\{k, l\} \in E(G)$ and $k \notin N(i)$ or $l \notin N(i)$; (ii) $\{k, l\} \notin E(G)$ and $k, l \in N(i)$. The mapping $\gamma_i(G) = G_i^c$ is called *local complementation* at i . Note that $\gamma_i(\gamma_i(G)) = \gamma_i(G_i^c) = G$. The *Clifford group* \mathcal{C}_1 on one qubit is the group of all 2×2 unitary matrices C , for which $C\sigma_u C^\dagger = \pm\sigma_{\pi(u)}$, where $u \in \{x, y, z\}$ and $\pi \in S_3$, the full symmetric group on three symbols. The generators of \mathcal{C}_1 are the matrices $\tau_x = \sqrt{-i\sigma_x}$ and $\tau_z = \sqrt{i\sigma_z}$. The *Clifford group* \mathcal{C}_n on n -qubits is the n -fold tensor product of elements of \mathcal{C}_1 . Two graph states $|G\rangle$ and $|H\rangle$ of dimension 2^n are said to be *LC-equivalent* if there is $U \in \mathcal{C}_n$ such that $U|G\rangle U^\dagger = |H\rangle$. It may be interesting to mention that the interlace polynomial is an invariant under local complementation [33]. Hein *et al.* [27] and Van den Nest *et al.* [40] proved the following link between local complementation and LC-equivalence: two graph states $|G\rangle$ and $|H\rangle$ are LC-equivalent if and only if $\gamma_k(\gamma_j(\dots(\gamma_i(G)))) = H$, where $\gamma_k, \gamma_j, \dots, \gamma_i$ is a sequence of local complementations at vertices $k, j, \dots, i \in V(G)$. Let G^I be the graph on n vertices whose adjacency matrix is obtained from $M(G)^{-1}$ by changing the sign of the negative entries. Jaeger proved that if $r(G) = n$ then $r(G^I) = n$ and G^I can be obtained from G by a sequence of local complementations [28] (see also [8]). This implies the following:

Proposition 7 *Let $|G\rangle$ be a two-colorable graph state with maximal Schmidt measure. Then $|G^I\rangle$ has maximal Schmidt measure. Moreover, $|G\rangle$ and $|G^I\rangle$ are LC-equivalent.*

B. Line graph

The *line graph* of a graph G , denoted by $L(G)$, is the graph whose set of vertices is $E(G)$ and $\{\{i, j\}, \{k, l\}\} \in E(L(G))$ if and only if one of the following conditions is satisfied: $j = k$, $j = l$, $i = k$ or $i = l$. The rank of the line graph of a graph on n vertices is at least $n - 2$ ([1], Proposition 14). A *clique* is an induced complete subgraph. In a graph G , a vertex i is a *cutpoint* if the graph $G \setminus i$, obtained by deleting i and all edges incident with i , has more connected components than G . Given a graph G , we have $G = L(T)$ for some tree T if and only if $E(G)$ can be partitioned into a set of cliques with the property that any vertex is in either one or two cliques; if a vertex is in two cliques then it is a cutpoint. Gutman

and Sciriha proved that if T is a tree then $L(T)$ is either nonsingular or it has nullity 1 ([25], Theorem 2.1). However, the line graph transformation applied to trees it is not directly relevant to our context, since we need $L(T)$ to be bipartite. In fact, it is clear that $L(T)$ is bipartite (and a path) if and only if T itself is a path.

C. Switching

Proposition 7 gives a method to obtain a graph state (not necessarily two-colorable) with maximal Schmidt measure from a two-colorable graph state with the same entanglement. The graph operation considered in Proposition 7 is then guaranteed to preserve the amount of entanglement (which is that case is maximal). Is there a simple operation that is guaranteed to change the amount of entanglement? *Switching* is a graph operation introduced by Van Lint and Seidel [39] (see [26], for a survey). The graph G_i^s is the *switching* of G at i if $V(G_i^s) = V(G)$ and $\{k, l\} \in E(G_i^s)$ if and only if one of the following two conditions is satisfied: (i) $\{k, l\} \in E(G)$ and, $k \neq j$ and $l \neq i$; (ii) $\{k, l\} \notin E(G)$ and, $k = i$ or $l = i$. The mapping $s_i(G) = G_i^s$ is called *switching* at i . It follows directly from the definition that $A(G_i^c) = A(G) - D + C$, where $D = \sum_{j \in N(i)} e_i e_j^T + \left(\sum_{j \in N(i)} e_i e_j^T\right)^T$ and $C = \sum_{j \notin N(i)} e_i e_j^T + \left(\sum_{j \notin N(i)} e_i e_j^T\right)^T$. Deciding if a graph G can be obtained from a graph H by switching is polynomial time equivalent to graph isomorphism [17]. The *switching operator* with respect to k is the unitary operator $T_k : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^n}$ defined as

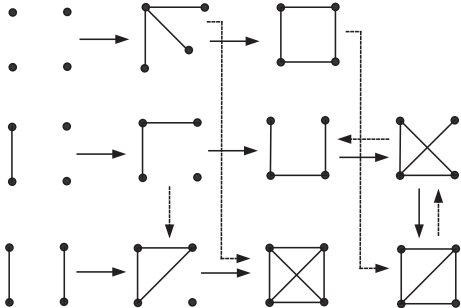
$$T_k |x_1 x_2 \dots x_n\rangle = (-1)^{x_k \sum_{i=1: i \neq k} x_i} |x_1 x_2 \dots x_n\rangle,$$

for $x_1, x_2, \dots, x_n \in \{0, 1\}$. It is easy to see that for the graph states $|G\rangle$ and $|G_k^s\rangle$, we have $T_k |G\rangle = |G_k^s\rangle$. A group \mathcal{G} acting (on the left) on a set Ω is *transitive* if for every $\alpha, \beta \in \Omega$ there is $g \in \mathcal{G}$ such that $g\alpha = \beta$. Let $\Omega_n = \{G : V(G) = \{1, 2, \dots, n\}\}$. Ehrenfeucht *et al.* [13] prove that the composition of local complementation and switching forms a transitive group acting on the set Ω_n . Let us denote by Ω_k^S the set of graph states of dimension k . We can then write:

Proposition 8 *The composition of elements from the local Clifford group and switching operators forms a transitive group acting on the set Ω_k^S .*

The meaning of this observation is clear: from a graph state $|G\rangle$ of a given dimension one can obtain any other graph state $|H\rangle$ of the same dimension by the application of elements from the local Clifford group and switching operators. The *signed adjacency matrix* of a graph G is the matrix $A^+(G)$ such that $A^+(G)_{i,j} = 1$ if $\{i, j\} \in E(G)$, $A^+(G)_{i,j} = -1$ if $\{i, j\} \notin E(G)$. The spectrum of $A^+(G)$ is equal to the spectrum of $A^+(H)$ if the graphs G and H are obtained one from the other one

by switching. As a consequence, the spectrum of $A^+(G)$ does not seem to contain much information about the entanglement properties of $|G\rangle$. In the figure below are drawn all (nonisomorphic) graphs on four vertices. An arrow between two graphs indicates that the graphs are in the same switching class; a dotted arrow indicates that the graphs are in the same local complementation class:



Notice that there are some graphs which are linked by both, local complementation and switching. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the

minimum number of colors needed to color the vertices of G such that adjacent vertices have different colors. Let $[G]$ be the *switching class* of G , that is the set of all graphs obtained by a sequence of switching on a graph G . If $\chi(G) = k$ then $2 \leq \chi(H) \leq 2k$, for all $H \in [G]$. If a switching class has a graph with chromatic number larger than 4 then it does not contain a bipartite graph, but the converse it is not necessarily true ([26], Lemma 3.30). It is not immediate to understand how entanglement is modified by switching. However, on the light of Proposition 8, switching is potentially useful in classifying graph states (see [11], for a work on the classification of graph states).

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- [1] S. Akbari, P. J. Cameron and G. B. Khosrovshahi, Ranks and signatures of adjacency matrices, submitted (2005).
 - [2] H. Aschauer, W. Dür and H.-J. Briegel, Multiparticle entanglement purification for two-colorable graph states, *Phys. Rev. A* **71**, 012319 (2005).
 - [3] J. H. Bevis, K. K. Blount, G. J. Davis, G. S. Domke, J. M. Lalani, V. A. Miller, Recent results involving the rank of the adjacency matrix of a graph, Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), *Congr. Numer.* 100 (1994), 33–45.
 - [4] J. H. Bevis, G. S. Domke, and V. A. Miller, Ranks of trees and grid graphs, *J. of Combinatorial Math. and Combinatorial Computing* **18** (1995), 109–119.
 - [5] N. L. Biggs, *Algebraic Graph Theory*, Cambridge Tracts in Mathematics **67**, Cambridge University Press 1974.
 - [6] T. Bıyıkoglu, Rank and Number of Nodal Domains of Cographs, submitted (2003).
 - [7] A. Bouchet, Transforming trees by successive local complementations, *J. Graph Theory* **12** (1988) 195–207.
 - [8] A. Bouchet, Bipartite graphs that are not circle graphs, *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 3, 809–814, Symposium à la Mémoire de François Jaeger (Grenoble, 1998).
 - [9] H.-J. Briegel and R. Raussendorf, *Phys. Rev. Lett.* **86**, 910 (2001).
 - [10] D. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs — Theory and Application*, Academic Press, New York, 1980.
 - [11] L. E. Danielsen, *On Self-Dual Quantum Codes, Graphs, and Boolean Functions*, Master's thesis, Department of Informatics, University of Bergen, Norway, March 2005. quant-ph/0503236.
 - [12] L. E. Danielsen and M. G. Parker, On the classification of all self-dual additive codes over $\text{GF}(4)$ of length up to 12, 2005. To appear in *Journal of Combinatorial Theory A*. math.CO/0504522.
 - [13] A. Ehrenfeucht, T. Harju, G. Rozenberg, Transitivity of local complementation and switching on graphs. *Discrete Math.* **278** (2004), no. 1-3, 45–60.
 - [14] J. Eisert and H.-J. Briegel, Schmidt measure as a tool for quantifying multiparticle entanglement, *Phys. Rev. A* **64**, 022306 (2001).
 - [15] D. de Caen, The ranks of tournament matrices, *Amer. Math. Monthly*, 98 (1991), 829–931.
 - [16] K. Chen, H.-K. Lo, Multi-partite quantum cryptographic protocols with noisy GHZ states. quant-ph/0404133.
 - [17] C. J. Colbourn, D. G. Corneil, On deciding switching equivalence of graphs, *Discrete Appl. Math.* **2** (1980), no. 3, 181–184.
 - [18] D. Fattal, T. S. Cubitt, Y. Yamamoto, S. Bravyi, I. L. Chuang, Entanglement in the stabilizer formalism. quant-ph/0406168.
 - [19] D. Fon-der-Flaas, On local complementations of graphs, in: A. Hajnal, L. Lovasz, V.T. Sos (Eds.), *Combinatorics (Eger, 1987)*, Colloquia Mathematica Societatis Janos Bolyai, Vol. 52, North-Holland, Amsterdam, 1988, pp. 257–266.
 - [20] G. H. Fricke, S. T. Hedetniemi, D. P. Jacobs, and V. Trevisan, Reducing the adjacency matrix of a tree. *Electron. J. Linear Algebra* **1** (1996), 34–43.
 - [21] M. C. Golumbic, A. N. Trenk, *Tolerance graphs*, Cambridge Studies in Advanced Mathematics, 89, Cambridge University Press, Cambridge, 2004.
 - [22] R. L. Graham and N. J. A. Sloane, Anti-Hadamard Matrices, *Linear Algebra and Applications*, **62** (1984), pp. 113–137.
 - [23] A. Graovac, I. Gutman, N. Trinajstić, *Topological Approach to the Chemistry of Conjugated Molecules*, Springer, Berlin, 1977.

- [24] M. Grassl, A. Klappenecker, and M. Rötteler, *Graphs, Quadratic Forms, and Quantum Codes*, in Proc. 2002 IEEE International Symposium on Information Theory, Lausanne, Switzerland (2002), page 45.
- [25] I. Gutman and I. Sciriha, On the nullity of line graphs of trees, *Discrete Mathematics* **232** (2001), 35-45.
- [26] J. Hage, Structural aspects of switching classes, Ph.D. Thesis, Leiden University, Leiden, 2001.
- [27] M. Hein, J. Eisert, and H.J. Briegel. Multi-party entanglement in graph states, *Phys. Rev. A* **69**, 062311 (2004).
- [28] F. Jaeger, Graphes de cordes et espaces graphiques, *Europ. J. Combin.*, **4** (1983), 319-217.
- [29] J. Kahn, J. Komlós, E. Szemerédi, On the probability that a random ± 1 matrix is singular, *J. Amer. Math. Soc.* **8** (1995), 223-240.
- [30] L. Lovasz, *Discrete Mathematics* **2** (1972) 253-267.
- [31] T. S. Michael, The Ranks of Tournament Matrices, *Amer. Math. Monthly*, 102 (1995), 637-639.
- [32] R. Raussendorf, D. E. Browne, and H. J. Briegel, *Phys. Rev. A* **68**, 022312 (2003).
- [33] C. Riera and M. G. Parker, One and Two-Variable Interlace Polynomials: A Spectral Interpretation, submitted to Proceedings of WCC2005, Bergen, Lecture Notes in Computer Science, LNCS, October 2005.
- [34] F. G. Royle, The Rank of a Cograph, *Electr. J. Comb.* **10** (2003), 2001.
- [35] D.-M. Schlingemann, Stabilizer codes can be realized as graph codes, *Quant. Inf. Comp.* **2**, 307 (2002)
- [36] D.-M. Schlingemann, Cluster states, graphs and algorithms, *Quant. Inf. Comp.* **4**, 287 (2004), quant-ph/0305170.
- [37] D. Seinsche, On a Property of the Class of n -Colorable Graphs, *Journal of Combinatorial Theory B* **16** (1974) 191-193.
- [38] Torsten Sillke, Graphs with maximal rank, www.mathematik.uni-bielefeld.de/~sillke
- [39] J. H. Van Lint, J. J. Seidel, Equilateral point sets in elliptic geometry, *Nederl. Akad. Wetensch. Proc. Ser. A* **69** (1966) 335-348.
- [40] M. Van den Nest, J. Dehaene, and B. De Moor, *Phys. Rev. A* **69**, 022316 (2004).
- [41] T. Voigt, G. M. Ziegler, Singular 0/1-matrices, and the hyperplanes spanned by random 0/1-vectors, math.CO/0308050.
- [42] T. Xuezhong, B. Liu, On the nullity of unicyclic graphs, *Linear Algebra Appl.* **408** (2005), 212-220.